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Fractional Fuzzy Adaptive Methodology for Fractional-order Non-Affine Nonlinear Systems: Application to Gyroscope

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***Abstract*— This study employs a fractional fuzzy adaptive methodology to design procedures for fractional-order non-affine nonlinear systems. The significant evolution of fractional-order calculus in science and engineering has made this area one of the most widespread fields, particularly in control engineering. Fractional-order fuzzy adaptive controller (FAC) has involved numerous scientists to improve appropriate controllers for non-affine nonlinear systems because of: 1) reconfigurable framework, the performance of the FAC is superior to that of the fuzzy controllers, 2) using the experts' data, FAC can apply the expert knowledge in the controller procedure rather than adaptive ones, and 3) enhancement of the controller routine instead of the integer-order one. In addition, this approach can control nominal systems in the presence of both external disturbances and uncertainties. The fractional-order adaptation laws are developed to guarantee the stability of the closed-loop system using a fractional-order Lyapunov approach. Unlike other research that focuses on fractional-order affine nonlinear systems, our approach specifically addresses fractional-order nonaffine nonlinear systems. Finally, the performance of the proposed methodology on chaotic systems, a gyroscope, and an inverted pendulum indicates the capability of the proposed scheme.**

Keywords— Non-Affine Nonlinear System, Adaptive Control, Fractional Order (FO) Systems, Fractional-Order Lyapunov Stability, Fuzzy System.

I. Introduction

Fractional Order Calculus (FOC), despite a history of over years in mathematics, has recently received much attention as a new subject in engineering and basic sciences. FOC, which has a greater degree of freedom than integer-order calculus by generalizing derivative and integral order to real numbers, has a variety of applications in electronics, telecommunications, control, mechanics, physics, and even medicine [4]. FOC has a noticeable advantage over integer order calculations, as is investigated in various studies for heat transfer process modeling [4], electrochemical processes [4], biological systems [5], diffusion procedure in batteries [6], dielectric polarization, viscoelastic systems, and electromagnetic waves [7].

Furthermore, fractional-order controllers have shown a more significant performance than integer-order controllers. For the first time, Oustaloup provided a new way for fractional-order calculations to enter control by introducing a robust fractional-order controller called commande robuste d'ordre non entier (CRONE) [8]. Consequently, many articles and research were presented to control fractional-order systems or introduce new procedures in fractional-order controllers. These include fractional-order Proportional-Integral-derivative (PID) controllers [9]-[10], fractional-order model reference controllers [11]-[12], and synchronization of chaotic fractional-order nonlinear systems [13]-[16]. Also, due to differences in the concept of energy in fractional and integer order systems, new ideas of stability of

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fractional order systems and Lyapunov theory have been introduced and considered in [17]-[18].

In [19], a novel terminal sliding mode observer is developed using neural networks for a nonlinear system. Moreover, authors in [20] deal with fractional nonsingular terminal sliding mode controllers for nonlinear fractional-order chaotic systems.

On the other hand, fuzzy systems are valuable for employing expert knowledge and have received considerable attention within the control field in the last two decades [1]. The importance of fuzzy systems lies in designing controllers based on experts' knowledge. Also, it has been proven that if the required conditions are met, fuzzy systems are comprehensive approximators and can estimate any continuous nonlinear function of any degree. They are also referred to as model-independent control methods. In addition, fuzzy logic has been able to open its place in adaptive control with different structures, and adaptive fuzzy control has shown promising performance.

Since fractional order calculus has recently developed in control systems, the combination of fractional order systems and adaptive fuzzy controllers is considered a novel field in intelligent control engineering and research. The first fractional order fuzzy adaptive controller was presented by Onder Efe in 2008 to guide a two-degree-of-freedom robot arm with an integer dynamic model [21]. The study [22] covers the sliding mode and indirect fuzzy adaptive controller to synchronize fractional-order nonlinear chaotic systems. Also in [23], the H^∞fuzzy adaptive controller is designed to synchronize the fractional-order nonlinear system. In [24], an adaptive fuzzy controller and sliding mode approach are used for fractional-order time-delayed nonlinear systems. However, the disadvantage of [22]-[24] is the lack of sufficient accuracy in fractional-order mathematical calculations, which makes the results unusable. The authors in [25]-[27] propose a fractional-order nonlinear system based on a hybrid fuzzy adaptive controller, which uses improper fractional-order calculus equations. Furthermore, [27] uses integer-based Lyapunov's theorem to prove fractional-order systems incorrectly.

In [30], an interval type-2 fuzzy adaptive controller is presented for both synchronization and stabilization of chaotic nonlinear fractional-order systems. In [31], a fuzzy adaptive controller is applied to synchronize and stabilize fractional-order nonlinear systems in the presence of uncertainties. Sliding mode control is a common technique to control fractional-order systems with uncertainty and external disturbances [34]-[35]. In [36], the adaptive fuzzy controller has been applied to fractional order uncertain systems in the presence of input constraints. The adaptive fuzzy approach can also be used for time-delayed systems [37]. Various adaptive methods have been proposed for fault tolerance in fractional-order systems [38]-[39]. Authors in [42] deal with a fuzzy adaptive consensus controller for a class of incommensurate fractional-order systems. The FO sliding mode controller is developed for a class of affine nonlinear systems in [43].

The primary disadvantages of the proposed method are two-fold:

1) Neither of the referenced studies examines fractional nonaffine nonlinear systems.

2) In designing an adaptive controller for an unknown nonlinear system, most sources rely on approximating the unknown functions of the system using a fuzzy system based on the Lyapunov theorem. This approach significantly increases the computational burden.

The present article proposes a fractional-order adaptive fuzzy strategy that controls uncertain nonlinear fractional-order systems in the presence of disturbances. The advantage of the proposed method is that it overcomes the uncertainty and external disturbance in the nonlinear fractional model. A Fuzzy system is considered to estimate the control input as a universal approximator and to apply the experts' knowledge in designing controller procedures. The closed-loop stability is guaranteed in the sense of Lyapunov.

The organization of the paper is explained as follows: a review of the preliminary concepts of fractional calculus and the fuzzy system is provided in Section 2. The problem statement is presented in Section 3. The design of fractional fuzzy adaptive control is proposed in Section 4. Section 5 is dedicated to illustrating the numerical simulations. Eventually, a brief summarization is presented in the last section.

II. PRELIMINARIES

The present section intends to discuss the pivotal preliminary, stability definitions, the fractional calculus, which relies on Mittag-Leffler theory, and fuzzy logic systems.

Fractional Calculus

For a better understanding of fractional-order systems, this section briefly examines these systems.

A critical role is associated with fractional calculus in recent contexts. The differential equations of fractional order are applied to describe the control system. The arbitrary orders of derivatives and integrals are indeed allowed by fractional calculus. The definition of a general calculus operator (comprising both fractional and integral orders) is presented as [40]:

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & q > 0 \\ 1, & q = 0 \\ \displaystyle\int_a^t (d\tau)^{-q}, & q < 0 \end{cases} $$

Where q and a are arbitrary real numbers. Below, the description of three common definitions for the fractional derivative and integral is presented.

Definition 1: [28] The q-order Grunwald - Letnikov (GL), Riemann-Liouville (RL), and Caputo(C) derivatives of the function $f(t)$ are described as:

$$ {}_a^{GL}D_t^q f(t) = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j q_j \; f\left(t - j\left[\frac{t-a}{N}\right]\right) $$

$$ {}_a^{RL}D_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_a^t (t-\tau)^{-q} f(\tau) d\tau $$

$$ {}_a^{C}D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-\tau)^{-q} \dot{f}(\tau) d\tau \quad (1) $$

where $0 < q < 1$ and $\Gamma(.)$ is the Gamma function.

Definition 2: [28] The q- order Riemann-Liouville fractional integral of $f(t)$ is defined as:

$$ {}_aD_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau. \quad (2) $$

Definition 3: [28] The Mittag-Leffler function for solving fractional order systems is defined likewise the exponential function applied to solving integer-order systems, as follows:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \tag{3}$$

where $\alpha > 0$.

The next equation explains the Mittag-Leffler function in terms of two parameters:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{4}$$

where $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_{\alpha,1}(z) = E_{\alpha}(z)$. Also $E_{1,1}(z) = e^z$.

According to various references, the most suitable definition for engineering applications is the Caputo (C) approach.

Definition 4: [28] (Mittag-Leffler Stability) The solution of ${}_0^C D_t^q x(t) = f(x,t)$ is considered to be Mittag-Leffler stable if

$$\| x(t) \| \leqslant \left\{ m[x(t_0)](t-t_0)^{-\gamma} E_{\alpha,1-\gamma}(-\lambda(t-t_0)^{\alpha}) \right\}^b \tag{5}$$

where $\alpha \in (0,1)$, $\gamma \in [0,1-\alpha]$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, $m(x)$ is locally Lipschitz on $x \in B \subseteq \mathbb{R}^n$ with Lipschitz constant, and t_0 is the initial time.

Theorem 1: [28] Let $x = 0$ be an equilibrium point for the system ${}_0^C D_t^q x(t) = f(x,t)$ and $D \subseteq \mathbb{R}$ be a domain containing zero. Let $V(t, x(t)) : [0,\infty) \times D \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to in order that

$$\alpha_1 \| x \|^a \leq V(t,x(t)) \leq \alpha_2 \| x \|^{ab}$$
$${}_0^C D_t^{\beta} V(t,x(t)) \leq -\alpha_3 \| x \|^{ab} \tag{6}$$

where $t \geq 0$, $x \in D$, $\beta \in (0,1)$, α_1, α_2, α_3, a and b are arbitrary positive constants. Then, the origin is Mittag-Leffler stable. In the case that assumptions hold globally on $\mathbb{R}^n$, it could be deduced that the origin is globally Mittag-Leffler stable.

The common Asymptotic stability is implied by Mittag-Leffler stability.

Lemma 1: [29] Suppose that $x(t) \in \mathbb{R}$ is a derivable and continuous function. Hence, at each time instant $t \geq t_0$ it is deduced that:

$$\frac{1}{2}\left({}_{t_0}^C D_t^q x^2(t)\right) \leq x(t)\, {}_{t_0}^C D_t^q x(t) \qquad \forall q \in (0,1) \tag{7}$$

The main tool in the present study includes Caputo fractional order operators.

Fuzzy System

As the proposed method uses fuzzy systems as a general approximator, this section briefly discusses these relationships.

The fuzzy logic system is concisely illustrated in Fig. 1 [1]. The primary configuration of this system consists of a fuzzifier, a defuzzifier, and an engine for fuzzy inferences. This engine is made up of some IF-THEN rules to create a $U = [U_1 \times U_2 \times \ldots \times U_n]$ to $\mathbb{R}$ mapping, in which $X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n$ denotes a linguistic input vector, and the output of the fuzzy logic system is denoted by the linguistic variable $y \in \mathbb{R}$ [1]. The l^{th} fuzzy rule is denoted by:

$$R^l : If\ x_1\ is\ F_1^l\ and\ x_n\ is\ F_n^l\ \ then\ y\ \ is\ B^l \tag{8}$$

where F_i^l and B^l are the labels of the input and output fuzzy sets, respectively. Suppose l to be the number of fuzzy IF-THEN rules and consider i as the number of inputs of the fuzzy logic system. The fuzzy system output value would equal the following equation by applying product inference, singleton fuzzification, and center average defuzzification:

$$y(x) = \frac{\sum_{l=1}^{M} y^l \prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{M} \prod_{i=1}^{n} \mu_{F_i^l}(x_i)} \tag{9}$$

where the membership function for the linguistic variable x_i is denoted by $\mu_{F_i^l}(x_i)$, and the crisp value y^l is the value of y corresponding to the maximum value of the Gaussian membership function μ_{B^l} [3]. Hence, Equation (9) can result in the following by applying the fuzzy basis function (FBF) [1]:

$y(x) = \theta^T \xi(x)$ (10)

where $\theta = [y^1, y^2, \ldots, y^M]^T$ is a parameter vector and $\xi(x) = [\xi^1(x), \xi^2(x), \ldots, \xi^M(x)]^T$ is the fuzzy basis functions set definable by:

$$\xi^l(x) = \frac{\prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{M} \prod_{i=1}^{n} \mu_{F_i^l}(x_i)}. \tag{11}$$

Equation (9) is regarded as a universal approximator in terms of a fuzzy system if its parameters are chosen properly [2].

Lemma 2: [32] Assume $f : \Omega \to R$ is Lipschitz continuous for each $x \in C^1(I; \Omega)$ and $\epsilon > 0$, there would be a fuzzy logic system of (10) in a way that $sup|f(x(t)) - \theta^T \xi(x)| \leqslant \epsilon$ holds.

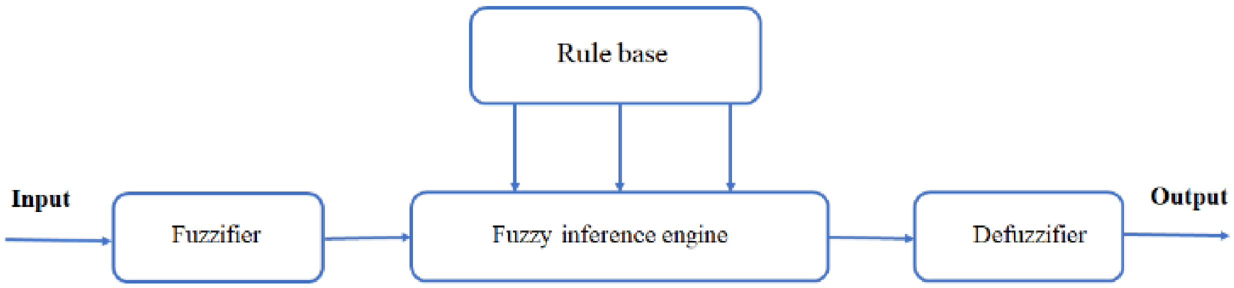

Fig. 1: Graph of the fuzzy logic system

Both Equation (11) and Lemma 2 are essential for understanding fuzzy systems as universal approximators.

III. PROBLEM STATEMENT

In the present study, the general form of the non-affine fractional-order system is considered below:

$$\begin{cases} D^q x_1 = x_2 \\ D^q x_2 = f(x,u) + \sigma(t) \\ y = x_1 \end{cases} \quad (12)$$

where $X = [x_1, x_2]^T$ is the state variable, $f(x,u)$ shows an unknown nonlinear function, $\sigma(t)$ is the bounded disturbance, and u presents the control input. The main control objective for the system in Equation (12) is to design a fuzzy adaptive fractional-order controller in a way that the system output $y(t)$ tracks a desired trajectory $y_d(t)$. In the meantime, all closed-loop system signals remain bounded. Concerning the system in Equation (12) and the desired trajectory, the next assumption should be considered.

Assumption 1: It is assumed that the nonzero function $f_u(x,u) = \frac{\partial f(x,u)}{\partial u}$ satisfies the following inequality, without loss of generality.

$$f_u(x,u) \geq F > 0 \quad (13)$$

in which $F \in \mathbb{R}$ is constant and known.

Assumption 2: The nonzero function $f_u(x,u)$ is supposed to fulfil the next inequality.

$$D^q\left(\frac{1}{f_u}\right) = -\frac{D^q f_u}{f_u} \leq 0 \quad (14)$$

Assumption 3: The boundary of the external disturbance is regarded as:

$$|\sigma(t)| \leqslant D. \quad (15)$$

Assumption 4: An arbitrary trajectory $y_d(t)$ as well as all its fractional time derivatives $D^q y_d(t)$ are bounded and smooth.

The tracking error is defined as:

$$e = [e_1, e_2]^T \quad (16)$$

where $e_1 = y_d - y$. By q-order time differentiating of Equation (16), we obtain:

$$\begin{cases} D^q e_1 &= D^q y_d - D^q y = D^q y_d - D^q x_1 \\ &= D^q y_d - x_2 = e_2 \\ D^q e_2 &= D^{2q} y_d - D^q x_2 \\ &= D^{2q} y_d - f(x,u) - \sigma(t) \end{cases} \quad (17)$$

So, Equation (17) can be rewritten as Equation (18).

$$D^q e = A_1 e + B[(D^{2q} y_d - f(x,u) - \sigma(t)] \quad (18)$$

where $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = [0 \quad 1]^T$.

Consider $A = A_1 - BK^T$ be Hurwitz, for $K = [k_1, k_2]^T$. Hence, according to the next equation, each symmetric positive definite matrix Q has a unique symmetric positive definite solution P such that [41]:

$$A^T P + PA = -Q \quad (19)$$

Let w be defined as

$$w = D^{2q} y_d + K^T e + \alpha tanh\left(\frac{B^T P e}{\varepsilon}\right) + w' \quad (20)$$

ε denotes a small positive constant, α denotes a large positive constant, and $tanh(.)$ is the hyperbolic tangent function, and w' stands for the adaptive term that describes later. If the term $K^T e + \alpha tanh(\frac{B^T P e}{\varepsilon}) + w'$ is added and subtracted to the right-hand side of Equation (18), it leads to:

$$D^q e = Ae - B[f(x,u) - w + \sigma(t) + \alpha tanh\left(\frac{B^T P e}{\varepsilon}\right) + w'] \quad (21)$$

Using Assumption 1, the next inequality is true for the signal, which is not explicitly dependent on the control input u:

$$\frac{\partial (f(x,u) - w)}{\partial u} = \frac{\partial f(x,u)}{\partial u} > 0 \quad (22)$$

Based on the implicit function theorem, the nonlinear algebraic equation $f(x,u) - w = 0$ is locally solvable for an arbitrary input (x,u). Hence, for any $(x,w) \in R^2 \times R$, some ideal controller $u^*(x,w)$ exists such that fulfills the next equality:

$$f(x,u^*) - w = 0 \quad (23)$$

Using the mean value theorem, a constant $\mu \in (0,1)$ exists in a way that the nonlinear function $f(x,u)$ is expressed around u^* as:

$$f(x,u) = f(x,u^*) + (u - u^*) f_{u_\mu} = f(x,u^*) + e_{u_\mu} f_{u_\mu} \quad (24)$$

where

$$\begin{cases} f_{u_\mu} = \frac{\partial (f(x,u))}{\partial u}\Big|_{u=u_\mu} \\ u_\mu = u\mu + (1-\mu) u^* \end{cases} \quad (25)$$

By replacing Equation (24) into the error Equation (21), we have:

$$D^q e = Ae - B[e_{u_\mu} f_{u_\mu} + \sigma(t) + \alpha tanh\left(\frac{B^T P e}{\varepsilon}\right) + w'] \quad (26)$$

Nonetheless, the existence of the ideal controller $u^*(x,w)$ for the system (23) is just assured according to the implicit function theory, but no solution technique is recommended yet. The next section deals with obtaining the unknown ideal control.

IV. FRACTIONAL ORDER FUZZY ADAPTIVE CONTROLLER

The previous section discusses the existence of an ideal controller for control objectives. This section focuses on developing a fuzzy system for the adaptive approximation of the unknown ideal controller. The ideal controller u^* is denoted as below.

$$u^* = f(z) + \epsilon \tag{27}$$

where $f(z) = \theta^{*T}\xi(z)$; θ^* and $\xi(z)$ are parameters and fuzzy basis functions, respectively. ϵ denotes an approximation error that fulfills $|\epsilon| \le \lambda$. Unknown parameter θ^* is calculated via the next optimization.

$$\theta^* = arg_\theta min[sup|\theta^T\xi(z) - f(z)|] \tag{28}$$

The approximation of θ^* is denoted by θ and u_{rob} is a robust controller used for compensating uncertainties, approximation error, interconnection term, and disturbance. The controller (27) is rewritten as:

$$u = \theta^T\xi(z) + u_{rob} \tag{29}$$

In which $\theta^T\xi(z)$ is the ideal controller approximation, and u_{rob} is defined below.

$$u_{rob} = \frac{|B^TPe|}{FB^TPe}(Fu_r + Fu_c + \hat{w}'). \tag{30}$$

where u_c is a compensation for uncertainties and approximation errors, $\hat{w}'$ estimates w, and u_r recompensates bounded external disturbances.

Using parameter error as $\hat{\theta} = \theta - \theta^*$, Equations (29) and (30), Equation (26) becomes:

$$D^q e = Ae - B[\hat{\theta}^T\xi(z) + u_{rob} - \epsilon)f_{u_\mu} + \sigma(t) + \alpha tanh(\frac{B^TPe}{\varepsilon}) + w'] \tag{31}$$

Consider the following update laws [20]:

$$\begin{cases} D^q\theta = \beta_1 B^TPe\xi(z) \\ D^q u_r = \beta_2|B^TPe| \\ D^q u_c = \beta_3|B^TPe| \\ D^q w' = \beta_4|B^TPe| \end{cases} \tag{32}$$

where $\beta_i > 0$, $i = 1, \ldots, 4$, are constant values. The updating laws will be derived later from the Lyapunov theorem, based on Theorem 2.

The block diagram of the overall controller is shown in Fig. 2.

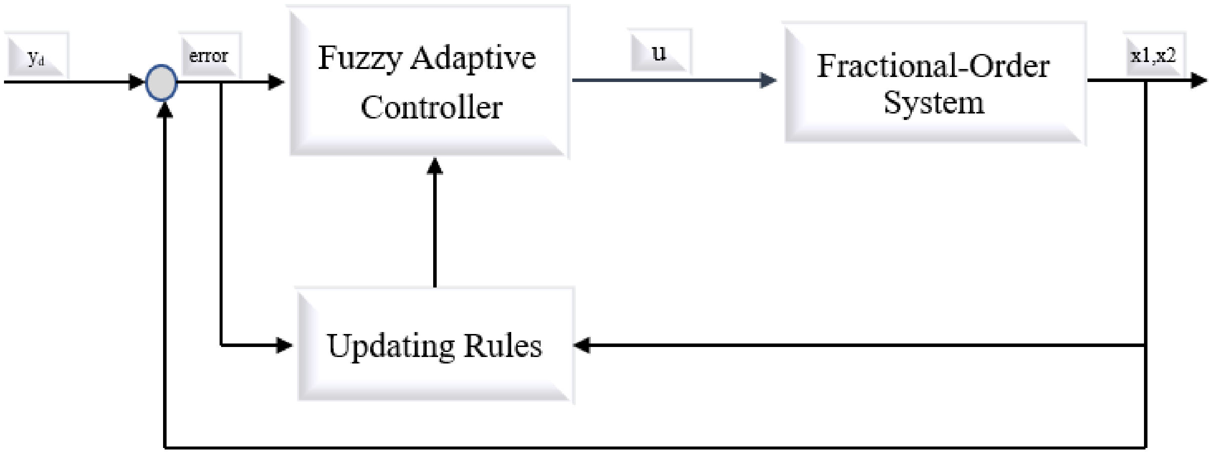

Fig. 2: The block diagram of the overall closed-loop system.

Theorem 2: Assume the error dynamical system of (31) and the external disturbances fulfilling Assumption 3. The controller structure described by Equations (29) and (30), along with the adaptation rules in Equation (32), ensures that the tracking error asymptotically converges to a neighborhood of zero and bounds all closed-loop system signals.

Proof: The following Lyapunov function is the starting point for proving the convergence of the tracking error, as well as the error of the parameters being bound to the neighborhood of the origin.

$$V = \frac{1}{2}[\frac{1}{f_u}e^TPe + \bar{\theta}^T\beta_1^{-1}\bar{\theta} + \frac{\bar{u}_r^2}{\beta_2} + \frac{\bar{u}_c^2}{\beta_3} + \frac{\bar{w}'^2}{\beta_4}] \tag{33}$$

where $\bar{\theta} = \theta - \theta^*$, $\bar{u}_r = u_r - \frac{D}{F}$, $\bar{u}_c = u_c - \epsilon$, and $\bar{w}' = w - w'$. According to Lemma 1, the q-order time derivative of the Lyapunov function becomes.

$$D^qV \le \frac{1}{2}[\frac{1}{f_u}(e^TPD^qe + D^qe^TPe) + D^q(\frac{1}{f_u})e^TPe] + \bar{\theta}^T\beta_1^{-1}D^q\bar{\theta} + \frac{\bar{u}_rD^q\bar{u}_r}{\beta_2} + \frac{\bar{u}_cD^q\bar{u}_c}{\beta_3} + \frac{\bar{w}'D^q\bar{w}'}{\beta_4} \tag{34}$$

By substituting the error dynamics Equation (31) into Equation (34), we derive the following results:

$$D^qV \le \frac{1}{2}[\frac{1}{f_u}e^T(A^TP + PA)e - (\frac{D^qf_u}{f_u})e^TPe] - \frac{1}{f_u}B^TPe[(\hat{\theta}^T\xi(z) + u_{rob} - \epsilon)f_{u_\mu} + \sigma(t) + \alpha tanh(\frac{B^TPe}{\varepsilon}) + w'] + \bar{\theta}^T\beta_1^{-1}D^q\bar{\theta} + \frac{\bar{u}_rD^q\bar{u}_r}{\beta_2} + \frac{\bar{u}_cD^q\bar{u}_c}{\beta_3} + \frac{\bar{w}'D^q\bar{w}'}{\beta_4} \tag{35}$$

Using the $B^TPe.tanh(\frac{B^TPe}{\varepsilon}) = |B^TPe|$ and Equation (19), Equation (35) can be rewritten as (36).

$$D^qV \leq \frac{1}{2}[-\frac{1}{f_u}e^TQe - \frac{D^qf_u}{f_u})e^TPe - \frac{\alpha}{f_u}|B^TPe|] + \frac{1}{f_u}B^TPe[(\hat{\theta}^T\xi(z) + u_{rob} - \epsilon)f_{u_\mu} + \sigma(t) + w' + \bar{\theta}^T\beta_1^{-1}D^q\bar{\theta} + \frac{\bar{u}_rD^q\bar{u}_r}{\beta_2} + \frac{\bar{u}_cD^q\bar{u}_c}{\beta_3} + \frac{\bar{w}'D^q\bar{w}'}{\beta_4} \quad (36)$$

Using Assumptions 1 and 3 and Equation (36), we have:

$$D^qV \leq \frac{1}{2}[-\frac{1}{f_u}e^TQe - \frac{D^qf_u}{f_u})e^TPe - \frac{\alpha}{f_u}|B^TPe|] + \frac{|B^TPe|}{F}|w'| - B^TPe\hat{\theta}^T\xi(z) - |B^TPe|u_{rob} + |B^TPe|\epsilon - \frac{|B^TPe|}{F}w' + \frac{|B^TPe|}{F}D + \bar{\theta}^T\beta_1^{-1}D^q\bar{\theta} + \frac{\bar{u}_rD^q\bar{u}_r}{\beta_2} + \frac{\bar{u}_cD^q\bar{u}_c}{\beta_3} + \frac{\bar{w}'D^q\bar{w}'}{\beta_4} \quad (37)$$

After performing several mathematical manipulations, the following equation is established.

$$D^qV \leq \frac{1}{2}[-\frac{1}{f_u}e^TQe - \frac{D^qf_u}{f_u})e^TPe - \frac{\alpha}{f_u}|B^TPe|] - B^TPe\hat{\theta}^T\xi(z) + \bar{\theta}^T\beta_1^{-1}D^q\bar{\theta} - |B^TPe|u_r + \frac{|B^TPe|}{F}D + \frac{\bar{u}_rD^q\bar{u}_r}{\beta_2} - |B^TPe|u_c + |B^TPe|\epsilon + \frac{\bar{u}_cD^q\bar{u}_c}{\beta_3} + \frac{|B^TPe|}{F}|w'| - \frac{|B^TPe|}{F}\hat{w} + \frac{\bar{w}'D^q\bar{w}'}{\beta_4} \quad (38)$$

By substituting the adaptive law described in Equation (32), the above inequality can be rewritten as follows:

$$D^qV \leq -\frac{1}{2}[\frac{1}{f_u}e^TQe + \frac{D^qf_u}{f_u}e^TPe + \frac{\alpha}{f_u}|B^TPe|] \quad (39)$$

Based on the proposed assumptions, $D^qV \leq 0$ is satisfied. Then, the origin is Mittag-Leffler stable and consequently, the equilibrium point of the system is asymptotically stable. Thus, according to Theorem 1, the tracking error converges to the neighborhood of the origin. Furthermore, all the signals involved in the closed-loop system are bounded. Thus, the proof is finalized.

V. SIMULATION RESULTS

In this section, a FO fuzzy adaptive controller is proposed, and three nonlinear FO examples are presented to validate the methodology's influence. In all cases mentioned below, the Caputo approach applies to the derivative operator.

Case A: *Chaotic System*

Consider the duffing dynamics discussed in the following FO nonlinear equations.

$$\begin{cases} D^qx_1 = x_2 \\ D^qx_2 = (1 + \mu sin(\omega t))x_1 - rx_2 - x_1^3 + u \end{cases} \quad (40)$$

where $\omega = 1$, $r = 0.2$, and $\mu = 1$. Five membership functions are defined for every input, and seven membership functions are considered for each output with 25 "If-Then" rules. Fig. s 3 and 4 show the phase portrait of the fractional-order duffing system.

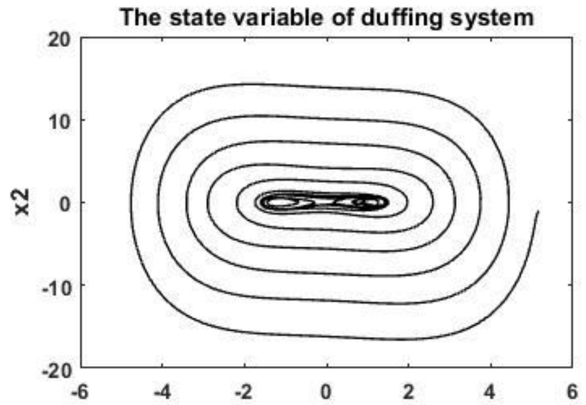

Fig. 3: The state variable of the duffing system with $[x_1(0), x_2(0)] = [5, -1]$.

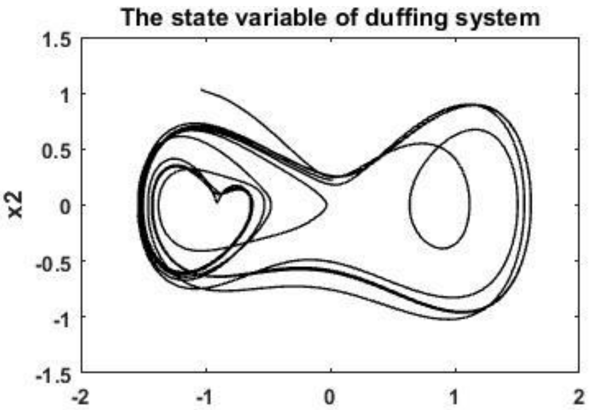

Fig. 4: The state variable of duffing system with $[x_1(0), x_2(0)] = [-1,1]$.

As it is obvious in Fig.s (3) and (4), the duffing chaotic system is sensitive to the initial values.

To reduce the sensitivity of the chaotic system to initial conditions and perform output tracking, we apply the planned controller, Equations (29) and (30), to the fractional-order Duffing system.

Fig. 5 presents the system's output and the desired trajectory under the planned controller.

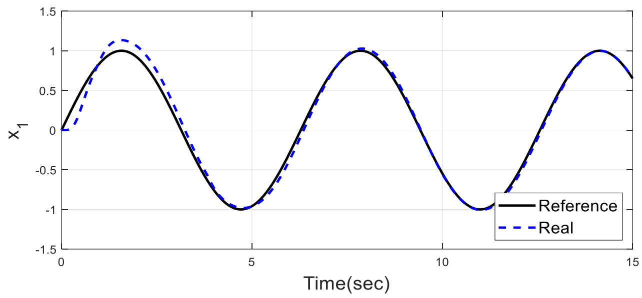

Fig. 5: Tracking trajectory of Case A

Fig. 6 depicts the tracking error as the difference between the system's output and its desired value.

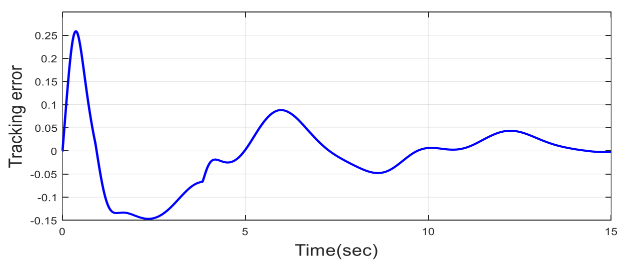

Fig. 6: Output Tracking error for case A

As shown in Fig. s 5 and 6, the performance of the proposed controller in the presence of external disturbances is satisfactory and furthermore denotes the convergence of tracking errors to zero in a considerably short time.

The control input of the system is depicted in Fig. 7.

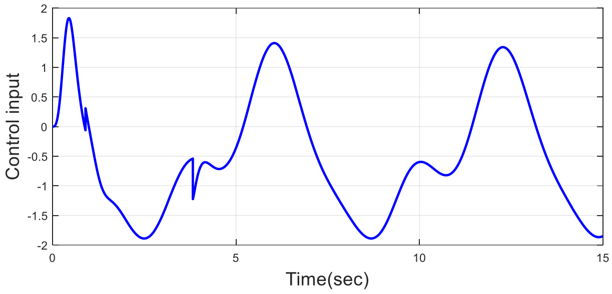

Figure 7: Control input for the FO Duffing system

As can be seen from Fig. s 5-7, it is evident that the proposed controller exhibits favorable performance. Furthermore, all signals involved in the closed-loop system are bounded in the above figures as proved by Theorem 2.

For comparing our approach with other methods, we apply a PID controller to the duffing system mentioned in Equation (40). The coefficients of the controller are tuned based on Ziegler-Nichols as $k_p = 2000, k_I = 100, k_D = 10$.

The reference signal and the system's output under the PID controller are depicted in Fig. 8.

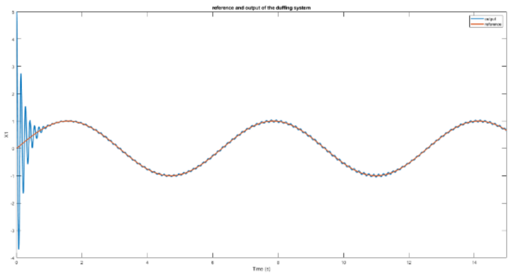

Fig. 8: Outputs of the system and reference signal under the PID controller.

The difference between the reference signal and the system's output is shown in Fig. 9.

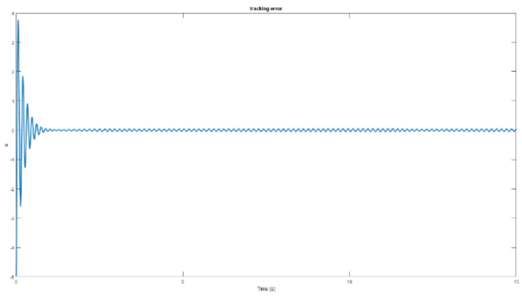

Fig. 9: Tracking error under the PID controller.

The control input of the system is illustrated in Fig. 10.

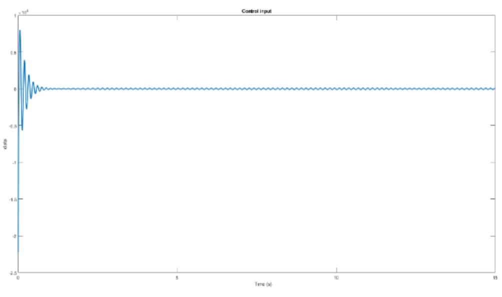

Fig. 10: Control input for the FO Duffing system under the PID controller

As shown in the above figures, it is clear that our approach outperforms the PID controller. However, the PID controller exhibits a faster convergence speed compared to our method. Moreover, the PID controller cannot eliminate the disturbances perfectly.

Case B: *Fractional order nonlinear Gyroscope*

To investigate the proposed methodology, we apply it to a fractional-order nonlinear gyroscope system. Thus, the fractional-order nonlinear nonaffine model of the gyroscope is:

$$D_t^q x_1 = x_2$$

$$D_t^q x_2 = -100\frac{\left(1-\cos\left(x_1^2(t)\right)\right)^2}{\sin^3(x_1(t))} - 0.5x_2(t) - 0.05x_2^3(t) + 35.5\sin(25t)\,\sin(x_1(t)) + \sin(x_1(t)) + \Delta_f(x,t) + d(t) + u(t) \quad (41)$$

where q stands for Caputo fractional order, x_1 is the rotation angle of a gyroscope, and x_2 shows the angular rotation speed. In our approach, the uncertainties and external disturbances are considered as [25]:

$$\begin{cases} \Delta_f = 4\sin(x_1) + \cos(x_2) \\ d(t) = 2\cos(2t) \end{cases} \quad (42)$$

Fig. 11 depicts the fractional-order nonlinear structure of the gyroscope [25].

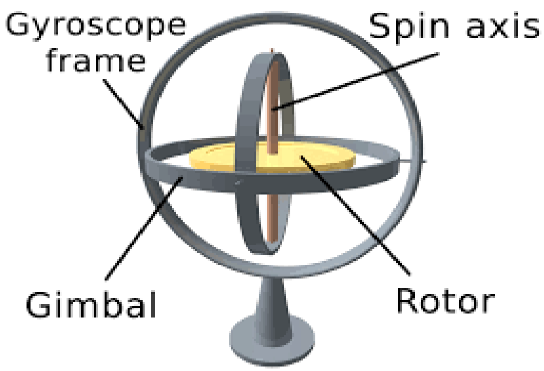

Fig. 11: Gyroscoper scheme [25]

In this case, five membership functions have been considered for input and seven for output. 25 "If-Then" rules are assigned for the fuzzy system.

Fig. 12 shows both the first state of the nonlinear fractional-order gyroscope controlled by a fuzzy adaptive controller, along with the reference signal.

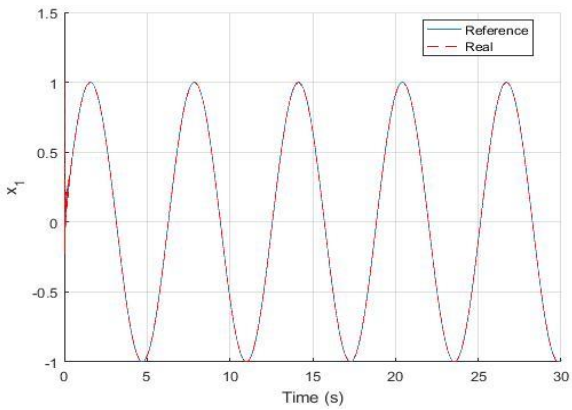

Fig.12: Tracking trajectory of gyroscoper system

The tracking error (difference between the first state of the nonlinear fractional-order gyroscope and the reference signal) is illustrated in Fig. 13.

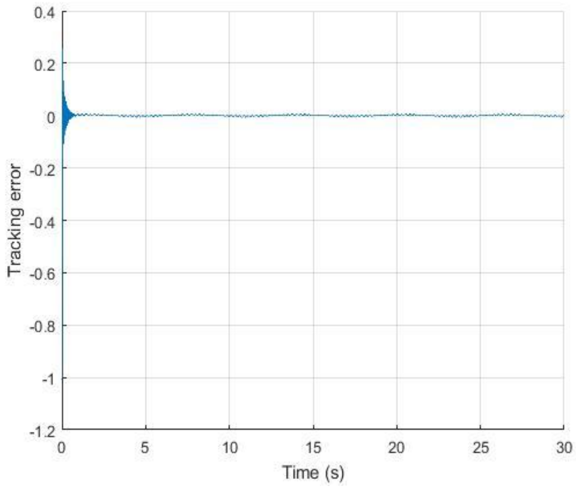

Fig.13: Tracking error of gyroscoper system.

As shown in Fig. s 12 and 13, the convergence of the tracking error to zero is guaranteed.

The control input of the gyroscope system (41) is demonstrated in Fig. 14.

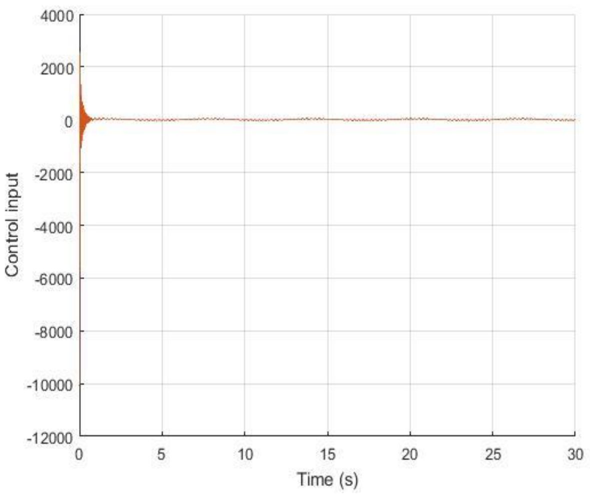

Fig. 14: Control signal of the gyroscoper system.

Based on Theorem 2 and Fig. 14, the control input is bounded. Fig. s 12- 14 show both the tracking error convergence to the origin and the boundedness of the signals of the closed-loop system, indicating that the proposed methodology has promising performance in the presence of external disturbances and uncertainties.

Case C: *Nonlinear inverted pendulum*

The planned policy is applied on a fractional-order nonlinear nonaffine inverted pendulum model, which is expressed below [33]:

$$aD_t^q x_{1=}x_2$$
$$aD_t^q x_2 = \frac{gsin(x_1)-mlx_2^2\cos(x_1)+\frac{\sin(x_1)}{m_c+m}}{\frac{4}{\frac{3lmcos2(x_1)}{m_c+m}}} + \frac{\frac{\cos(x_1)}{m_c+m}}{l\left[\frac{4}{\frac{3mcos2(x_1)}{m_c+m}}\right]}u + d(t) \quad (43)$$

where x_1 and x_2 respectively indicate the swing angle and swing speed, d(t) as the external disturbance. This approach considers five membership functions for input and seven for output. 25 "If-Then" rules are designated for fuzzy systems. Fig. 15 shows the inverted pendulum scheme [33].

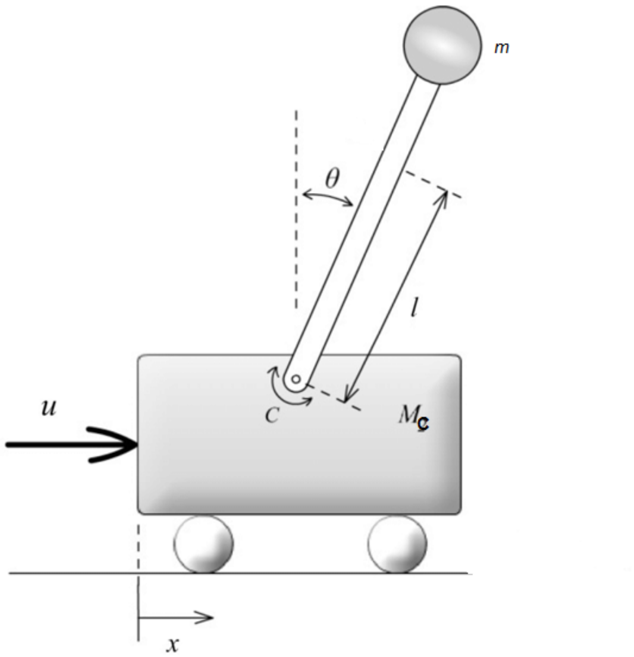

Fig. 15: Fractional order nonlinear Inverted Pendulum [33]

Table I summarizes the inverted pendulum parameters.

TABALE I
The Parameters of the Fractional-order Nonlinear Inverted Pendulum.

Symbols	Value	Unit
$\bar{g}$	gravitational acceleration	9.8 m/s^2
m_c	mass of the cart	1 Kg
m	mass of the pendulum	0.1 Kg
l	length to pendulum center of mass	0.5 m

The objective of control denotes the design of a fixed-time law in a way that the inverted pendulum motion tracks the assumed bounded reference trajectory as $r(t)$.

Both the tracking trajectory of the inverted pendulum using fuzzy adaptive methodology and the reference signal are shown in Fig. 16.

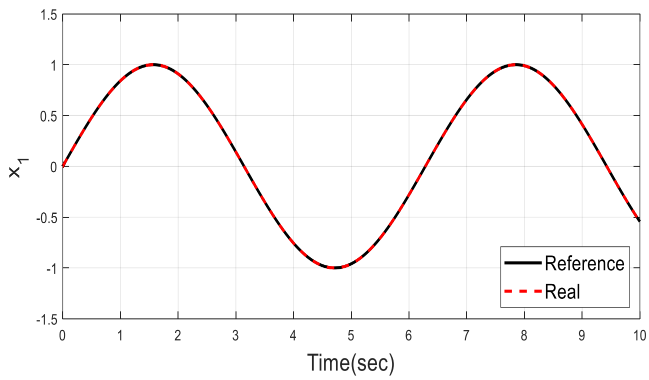

Fig.16: Tracking trajectory of the inverted pendulum.

The difference between the first state of (43) and the reference signal is presented in Fig. 17.

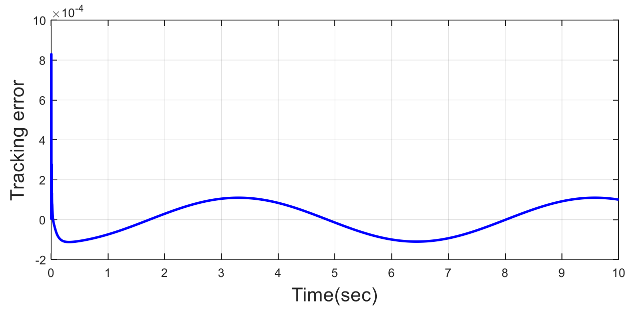

Fig. 17: Tracking error of the inverted pendulum

Fig. s 16 and 17 demonstrate that the system's output converges to the reference signal. Furthermore, the presence of uncertainties and disturbances can not affect the performance of the system. The control input is established in Fig. 18.

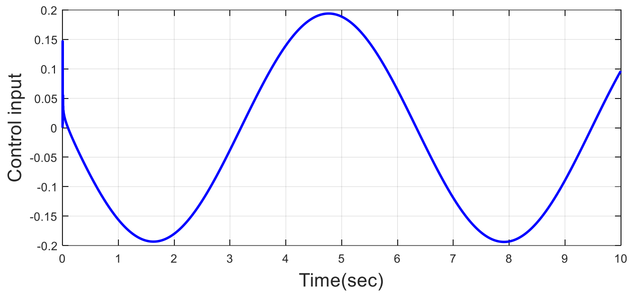

Fig. 18: Control input of the inverted pendulum

Fig. s 16 through 18 confirm the encouraging performance of the planned procedure.

This section clearly indicates that:

1- Convergence of the tracking error to zero is guaranteed.
2- Robustness against uncertainties and disturbances is guaranteed.
3- Boundedness of the signal involved in the closed-loop system is assured.

VI. CONCLUSION

This paper discusses a fuzzy adaptive output tracking methodology for fractional-order (FO) nonaffine nonlinear systems. Fuzzy logic is employed as a universal approximator, leveraging expert knowledge in the controller design process. The adaptation laws proposed in this study ensure both closed-loop stability, in accordance with Lyapunov's stability criteria, and the asymptotic convergence of the tracking error to a neighborhood around the origin. The main advantages of this methodology are: 1) the incorporation of expert knowledge in the controller design, and 2) robustness against disturbances, approximation errors, and model uncertainties. To demonstrate the effectiveness and performance of the proposed approach, it has been applied to three types of FO nonaffine nonlinear systems. Simulation results indicate that the method achieves an acceptable tracking error and a rapid response time. The main limitations of the proposed method are 1) applying Assumption 2 on the unknown function of the systems and 2) the complex controller structure in Section 3. The complexity of the proposed controller in sections 3 through 5 is due to both the structure and the unknown function of the system. In future work, we plan to explore the application of this methodology to real-world FO nonaffine nonlinear systems with input constraints.

REFERENCES

[1] LX. Wang, "A Course in Fuzzy Systems and Control", Prentice Hall PTR, 1997.

[2] LX. Wang, JM. Mendel, "Fuzzy basis function, universal approximation, and orthogonal least square learning", IEEE Trans Neural Network 3(5), 807–814, 1993.

[3] HJ. Zimmermann, "Fuzzy Set Theory and Its Application, Kluwer Academic Publishers, 1996.

[4] B. Vinagre and V. Feliu, " Modeling and control of dynamic systems using fractional calculus: Application to electrochemical processes and flexible structures", in Proc. 41st IEEE Decision and Control Conference, 214-239, 2002.

[5] R. L. Magin, "Fractional calculus in bioengineering", Begell House, Redding, publisher, 2006.

[6] J. Sabatier, M. Aoun, A. Oustaloup, G. Grégoire, F. Ragot, and P. Roy, "Fractional system identification for lead acid battery state of charge estimation", Signal processing, vol. 86, pp. 2645- 2657, 2006.

[7] R. Hilfer, "Applications of fractional calculus in physics", World Scientific publisher, 2000.

[8] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu-Batlle, "Fractional-order systems and controls: fundamentals and applications", Springer Science & Business Media publisher, 2010.

[9] A. Oustaloup, X. Moreau, and M. Nouillant, "The CRONE suspension," Control Engineering Practice 4(1), 1101-1108, 1996.

[10] M. Zamani, M. Karimi-Ghartemani, N. Sadati, and M. Parniani, "Design of a fractional order PID controller for an AVR using particle swarm optimization," Control Engineering Practice 17(1), 1380-1387, 2009.

[11] N. Aguila-Camacho and M. A. Duarte-Mermoud, "Fractional adaptive control for an automatic voltage regulator," ISA transactions 52(1), 807-815, 2013.

[12] Y. He and R. Gong, "Application of fractional-order model reference adaptive control on industry boiler burning system," in 2010 International Conference on Intelligent Computation, Changsha, China, 2010.

[13] L. Jun-Guo, "Chaotic dynamics and synchronization of fractional-order Genesio–Tesi systems," Chinese Physics 14(8), 1517, 2005

[14] C. Li and G. Chen, "Chaos and hyperchaos in the fractional-order Rössler equations," Physica A: Statistical Mechanics and its Applications 31(3), 55-61, 2004.

[15] J. G. Lu, "Chaotic dynamics of the fractional-order Lü system and its synchronization," Physics Letters A 354(1), 305-311, 2006.

[16] Q. Yang and C. Zeng, "Chaos in fractional conjugate Lorenz system and its scaling attractors," Communications in Nonlinear Science and Numerical Simulation 15(2), 4041-4051, 2010.

[17] MR. Rahimi Khoygani, R. Ghasemi, "Neural estimation using a stable discrete-time MLP observer for a class of discrete-time uncertain nonlinear systems", Nonlinear Dynamics 84(4), 2517-2533, 2016.

[18] MR. Rahimi Khoygani, R. Ghasemi, A.R. Vali, "Intelligent nonlinear observer design for a class of nonlinear discrete-time flexible joint robot", Intelligent Service Robotics 8(1), 45–56, 2015.

[19] A. Sharafian, R. Ghasemi, "A novel terminal sliding mode observer with RBF neural network for a class of nonlinear systems", International Journal of Systems, Control and Communications 9(4), 369-385, 2019.

[20] F. Shahbazi, R. Ghasemi, M. Mahmoodi, "Fractional Nonsingular Terminal Sliding Mode Controller Design for the Special Class of Nonlinear Fractional-order Chaotic Systems", International Journal of Smart Electrical Engineering 11(2), 83-88, 2021.

[21] M. Ö. Efe, "Fractional fuzzy adaptive sliding mode control of a 2-DOF direct-drive robot arm", Systems, IEEE Transactions on Man and Cybernetics Part B: Cybernetics 38(1), 1561-1570, 2008.

[22] T.-C. Lin, T.-Y. Lee, and V. E. Balas, "Adaptive fuzzy sliding mode control for synchronization of uncertain fractional order chaotic systems," Chaos, Solitons & Fractals 44(2), 791-801, 2011.

[23] T.-C. Lin, C.-H. Kuo, T.-Y. Lee, and V. E. Balas, "Adaptive fuzzy H∞ tracking design of SISO uncertain nonlinear fractional order time-delay systems," Nonlinear Dynamics 69(3), 1639- 1650, 2012.

[24] T.-C. Lin and T.Y. Lee, "Chaos synchronization of uncertain fractional-order chaotic systems with time delay based on adaptive fuzzy sliding mode control," IEEE Transactions on Fuzzy Systems 19(2), 623-635, 2011.

[25] M. P. Aghababa, Comments on "Adaptive fuzzy H∞ tracking design of SISO uncertain nonlinear fractional order time-delay systems", [Nonlinear Dyn. 69 (2012) 1639–1650]. *Nonlinear Dynamics*, *70*(4), 2511-2513, 2012.

[26] S. Tong, H.-X. Li, and W. Wei, "Comments on Direct adaptive fuzzy-neural control with state observer and supervisory controller for

unknown nonlinear dynamical systems", IEEE Transactions on Fuzzy Systems 11(2), 703-705, 2003.

[27] T.-C. Lin, C.-H. Kuo, and V. E. Balas, "Uncertain fractional order chaotic systems tracking design via adaptive hybrid fuzzy sliding mode control," International Journal of Computers Communications & Control 6(2), 418-427, 2011.

[28] Y. Li, YQ. Chen, I. Podlubny, "Mittag–Leffler stability of fractional-order non-linear dynamic systems", Automatica 45(8), 1965-1969, 2009.

[29] N. Aguila-Camacho, M. Duarte-Mermoud, J. Gallegos, "Lyapunov Functions for Fractional Order Systems", Communications in Nonlinear Science and Numerical Simulation 19(9), 2951-2957, 2014.

[30] P. Jafari, M. Teshnehlab, and M. Tavakoli-Kakhki, "Adaptive Type-2 Fuzzy System for Synchronization and Stabilization of Chaotic Nonlinear Fractional Order Systems," IET Control Theory & Applications 12(3), 183-193, 2017.

[31] P. Jafari, M. Teshnehlab, and M. Tavakoli-Kakhki, "Synchronization and stabilization of fractional order nonlinear systems with adaptive fuzzy controller and compensation signal," Nonlinear Dynamics 90, 1037-1052, 2017.

[32] L.X. Wang, "Adaptive Fuzzy Systems and Control: Design and Stability Analysis", Prentice-Hall, Inc., 1994.

[33] A. Kumar, V. Kumar, “An interval type-2 fractional order fuzzy logic controller employed to an uncertain nonlinear inverted pendulum”, 14th IEEE India Council International Conference, 2017.

[34] L. Chen, J. Fang, “Adaptive continuous sliding mode control for fractional order systems with uncertainties and unknown control gain”, Automation and Systems 20, 1509- 1520, 2022.

[35] N. Ullah, S. Han, MI. Khattak, “Adaptive fuzzy fractional order sliding mode controller for a class of dynamical systems with uncertainty”, Transactions of the Institute of Measurement and Control 38, 402- 413, 2016.

[36] S. Mirzajani, MP. Aghababa, A. Heydari, “Adaptive T- S fuzzy control design for fractional order systems with parametric uncertainty and input constraint”, Fuzzy Sets and Systems 365, 22- 39, 2019.

[37] X. Qin, S. Liu, “Adaptive fuzzy synchronization of uncertain fractional order chaotic systems with different structures and time delays”, Advances in Difference Equations 174, 2019.

[38] X. Hu, Q. Song, M. Ge, R. Li, “Fractional order adaptive fault tolerant control for a class of general nonlinear systems”, Nonlinear Dynamics 101, 379- 392, 2020.

[39] YX. Li, QY. Wang, S. Tong, “Fuzzy adaptive fault tolerant control of fractional order nonlinear systems”, IEEE Transactions on Systems, Man, and Cybernetics: Systems 51, 1372- 1379, 2021.

[40] A. Mohammadzadeh, O. Kaynak, “A novel fractional order fuzzy control method based on immersion and invariance approach”, Applied Soft Computing 88, 2020.

[41] O. Mofid, S. Mobayen, “Adaptive synchronization of fractional order quadratic chaotic flows with nonhyperbolic equilibrium”, Journal of vibration and control 24, 2018.

[42] A. Sharafian, A. Ali, I. Ullah, T.R. Khalifa, X. Bai, L. Qiu, “Fuzzy adaptive control for consensus tracking in multiagent systems with incommensurate fractional-order dynamics: Application to power systems”, Information Sciences 689(1): 121455, 2025.

[43] M.A.Z. Tajrishi, A. Akbarzadeh Kalat, “Fast finite time fractional-order robust-adaptive sliding mode control of nonlinear systems with unknown dynamics”, Journal of Computational and Applied Mathematics 438: 115554, 2024.